Lie families: theory and applications

José F. Cariñena[†], Janusz Grabowski[‡] and Javier de Lucas[‡]

[†]Departamento de Física Teórica, Universidad de Zaragoza,

50009 Zaragoza, Spain.

[‡]Institute of Mathematics, Polish Academy of Sciences,

ul. Śniadeckich 8, P.O. Box 21, 00-956 Warszawa, Poland

Abstract

We analyze families of non-autonomous systems of first-order ordinary differential equations admitting a *common time-dependent superposition rule*, i.e., a time-dependent map expressing any solution of each of these systems in terms of a generic set of particular solutions of the system and some constants. We next study relations of these families, called *Lie families*, with the theory of Lie and quasi-Lie systems and apply our theory to provide common time-dependent superposition rules for certain Lie families.

MSC 2000: *34A26, 34A05, 34A34, 17B66, 22E70*

PACS numbers: *02.30.Hq, 02.30.Jr, 02.40.-k*

Key words: *superposition rules, Lie–Scheffers systems, quasi-Lie schemes, Milne–Pinney equations, Abel equations.*

1 Introduction.

The theory of Lie systems [1, 2, 3, 4, 5, 6, 7] deals with non-autonomous systems of first-order ordinary differential equations such that all their solutions can be written in terms of generic sets of particular solutions and some constants, by means of a time-independent function. Such functions are called *superposition rules* and the systems admitting this mathematical property are called *Lie systems*. Lie succeeded in characterizing systems admitting a superposition rule. His result, known now as *Lie Theorem* [1], states that a non-autonomous system (time-dependent vector field) X_t is a Lie system if and only if there exists a finite-dimensional Lie algebra of vector fields V_0 such that $X_t \in V_0$ for all t.

Note that a superposition rule can be found explicitly even for systems whose general solution is not known, like in the case of Riccati equations [8], and its knowledge enables us to obtain the general solution out of certain sets of particular solutions in an easier way than solving directly the system.

In the theory of Lie systems various methods have been developed to obtain superposition rules, time-dependent and time-independent constants of the motion, exact solutions, integrability conditions, and other interesting properties for particular systems [9, 10, 11, 12, 13, 14, 15]. Unfortunately, being a Lie system is rather exceptional and, in order to apply the methods of the theory of Lie systems to a broader set of non-autonomous systems, some generalizations of this theory have been proposed. The generalized methods are presently used to investigate some partial differential equations [7], a class of second-order differential equations (the so-called *SODE Lie systems* [10]), certain Schrödinger equations [16], etc.

With the same aim of applying the theory of Lie systems to a broader family of systems, it has been recently developed the theory of *quasi-Lie schemes* and *quasi-Lie systems* [17, 18, 19]. This theory allows us to investigate some non-Lie systems and it can be applied to dealing with certain second- and even higher-order systems of differential equations. For example, it enables us to analyze some non-linear oscillators [17], dissipative Milne–Pinney equations [18], Emden-Fowler equations [19], etc. One of the main results obtained through *quasi-Lie scheme* approach is the existence of the so-called *time-dependent superposition rules*, that is, time-dependent superposition functions expressing the general solution in terms of a generic family of particular solutions of this system.

Note however that the concept of time-dependent superposition rule does not make much sense for a single non-autonomous system. This is because, as explained in [17], any single non-autonomous system admits such a superposition rule which, however, can be as difficult to finding as the general solution of the system and, therefore, it cannot be generally used to analyze properties of the system. This is analogous to the fact that each autonomous system is automatically a Lie system, as each single vector field spans a one-dimensional Lie algebra. Therefore, it only makes non-trivial sense to speak about common time-dependent superposition rules for a bigger family of non-autonomous systems.

In this paper, we give a natural generalization of Lie Theorem characterizing Lie systems. This enables us to show that many families of non-autonomous systems of first-order ordinary differential equations are *Lie families*, that is they admit common time-dependent superposition rules. Furthermore, we study some Lie families and we obtain common time-dependent superposition rules for all of them.

The organization of the paper goes as follows. Section 2 describes common time-dependent superposition rules in terms of certain horizontal foliations. In Section 3 we generalize Lie Theorem to characterize families of systems admitting common time-dependent superposition rules. We posteriorly use this result to analyze the relations between quasi-Lie systems, Lie systems and time-dependent superposition rules in Section 4. We finally apply all our results to investigate some Lie families throughout Section 5.

2 Time-dependent superpositions and foliations.

In this Section we develop the concept of common time-dependent superposition rule for a family of non-autonomous systems of first-order ordinary differential equations and relate this concept to certain horizontal foliations. For the sake of simplicity, we investigate these concepts in local coordinates, but our approach can be slightly modified to handle systems on manifolds.

Consider a family, parametrized by elements α of a set Λ, of non-autonomous systems of first-order ordinary differential equations on $\mathbb{R}^n$ of the form

$$\frac{dx^i}{dt} = Y^i_\alpha(t,x), \qquad i = 1,\ldots,n, \qquad \alpha \in \Lambda. \tag{1}$$

In applications, Λ is often a finite subset of $\mathbb{N}$ or $\Lambda = C^\infty(\mathbb{R})$. Solutions of these systems are integral curves of the family $\{Y_\alpha\}_{\alpha\in\Lambda}$ of time-dependent vector fields on $\mathbb{R}^n$ given by

$$Y_\alpha(t,x) = \sum_{i=1}^n Y^i_\alpha(t,x)\frac{\partial}{\partial x^i}, \qquad \alpha \in \Lambda. \tag{2}$$

Note 1. In order to simplify the terminology, we will use Y_α to designate both: a time-dependent vector field of the above family and the non-autonomous system describing its integral curves.

Denote with $\bar{Y}_\alpha$ the autonomization of the time-dependent vector field Y_α, that is, the vector field on $\mathbb{R}\times\mathbb{R}^n$ defined by

$$\bar{Y}_\alpha(t,x) = \frac{\partial}{\partial t} + \sum_{i=1}^n Y^i_\alpha(t,x)\frac{\partial}{\partial x^i}.$$

Integral curves of (1) can be identified with trajectories of the vector field (autonomous system) $\bar{Y}_\alpha$. Let us state the fundamental concept studied throughout the paper.

Definition 2. We say that the family of non-autonomous systems (1) admits a *common time-dependent superposition rule*, if there exists a map $\Phi : \mathbb{R}\times\mathbb{R}^{n(m+1)} \to \mathbb{R}^n$,

$$x = \Phi(t, x_{(1)},\ldots,x_{(m)}; k_1,\ldots,k_n), \tag{3}$$

such that the general solution $x(t)$ of any system Y_α of the family (1) can be written, at least for sufficiently small t, as

$$x(t) = \Phi(t, x_{(1)}(t),\ldots,x_{(m)}(t); k_1,\ldots,k_n),$$

with $\{x_{(a)}(t)\,|\, a = 1,\ldots,m\}$ being a generic set of particular solutions of Y_α, and $k_1,\ldots,k_n$ being constants associated with each particular solution. A family of systems (1) admitting a common time-dependent superposition rule is called a *Lie family*.

Note 3. We do not want to formalize precisely what 'generic' means in the above definition, as it is not crucial for our purposes and depends on the context. One can have in mind the following example: for a system of linear homogeneous differential equations 'generic' means that the particular solutions are linearly independent.

Given a common time-dependent superposition rule $\Phi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ of a Lie family $\{Y_\alpha\}_{\alpha\in\Lambda}$, the map $\Phi(t, x_{(1)}, \ldots, x_{(m)}; \cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x_{(0)} = \Phi(t, x_{(1)}, \ldots, x_{(m)}; k)$, is regular for a generic point $(t, x_{(1)}, \ldots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{nm}$ and, in view of the Implicit Function Theorem, it can be inverted to write

$$k = \Psi(t, x_{(0)}, \ldots, x_{(m)}),$$

for a certain map $\Psi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$, and $k = (k_1, \ldots, k_n)$ being the only point in $\mathbb{R}^n$ such that

$$x_{(0)} = \Phi(t, x_{(1)}, \ldots, x_{(m)}; k).$$

Note 4. As a matter of fact, the maps Ψ and Φ are defined only locally on open subsets of $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ but, for simplicity, we will write $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ for their domains.

Consequently, the map Ψ determines locally a n-codimensional foliation $\mathfrak{F}$ of the manifold $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ into the level sets of Ψ. Moreover, as the fundamental property of the map Ψ establishes that $\Psi(t, x_{(0)}(t), \ldots, x_{(m)}(t))$ is constant for any $(m+1)$-tuple of particular solutions of any system of the family (1), the foliation determined by Ψ is invariant under the permutation of its $(m+1)$ arguments $\{x_{(a)} \mid a = 0, \ldots, m\}$, and, differentiating $\Psi(t, x_{(0)}(t), \ldots, x_{(m)}(t))$ with respect to t, we get

$$\frac{\partial \Psi^j}{\partial t} + \sum_{a=0}^{m} \sum_{i=1}^{n} Y^i_\alpha(t, x_{(a)}(t)) \frac{\partial \Psi^j}{\partial x^i_{(a)}} = 0, \qquad j = 1, \ldots, n, \quad \alpha \in \Lambda, \tag{4}$$

where $\Psi = (\Psi^1, \ldots, \Psi^n)$.

Definition 5. Given a time-dependent vector field $Y = \sum_{i=1}^{n} Y^i(t, x)\partial/\partial x^i$ on $\mathbb{R}^n$, we define its *prolongation* to $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ as the vector field on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ given by

$$\widehat{Y}(t, x_{(0)}, \ldots, x_{(m)}) = \sum_{a=0}^{m} \sum_{i=1}^{n} Y^i(t, x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}, \tag{5}$$

and its *time-prolongation* to $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ as the vector field on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ of the form

$$\widetilde{Y}(t, x_{(0)}, \ldots, x_{(m)}) = \frac{\partial}{\partial t} + \sum_{a=0}^{m} \sum_{i=1}^{n} Y^i(t, x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}.$$

The equalities (4) show that the functions $\{\Psi^i \mid i = 1, \ldots, n\}$ are first-integrals for the vector fields $\{\widetilde{Y}_\alpha\}_{\alpha\in\Lambda}$, that is, $\widetilde{Y}_\alpha \Psi^i = 0$ for $i = 1, \ldots, n$ and $\alpha \in \Lambda$. Therefore, the vector fields $\widetilde{Y}_\alpha$ are tangent to the leaves of $\mathfrak{F}$.

The foliation $\mathfrak{F}$ has another important property. If the leaf $\mathfrak{F}_k$ is the level set of Ψ corresponding to a certain $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$, and given $(t, x_{(1)}, \ldots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{mn}$, there is only one point $x_{(0)} \in \mathbb{R}^n$ such that $(t, x_{(0)}, x_{(1)}, \ldots, x_{(m)}) \in \mathfrak{F}_k$. Thus, the projection onto the last $m \cdot n$ coordinates and the time

$$\pi : (t, x_{(0)}, \ldots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{n(m+1)} \longrightarrow (t, x_{(1)}, \ldots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{nm},$$

induces a local diffeomorphism from the leaf $\mathfrak{F}_k$ of $\mathfrak{F}$ into $\mathbb{R} \times \mathbb{R}^{nm}$. We will say that the foliation $\mathfrak{F}$ is *horizontal* with respect to the projection π.

On the other hand, the horizontal foliation defines the common time-dependent superposition rule without referring to the map Ψ. Indeed, if we take a point $x_{(0)}$ and m particular solutions, $x_{(1)}(t), \ldots, x_{(m)}(t)$, for a system of the family, then $x_{(0)}(t)$ is the unique curve in $\mathbb{R}^n$ such that the points of the curve

$$(t, x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) \subset \mathbb{R} \times \mathbb{R}^{nm}$$

belong to the same leaf as the point $(0, x_{(0)}(0), x_{(1)}(0), \ldots, x_{(m)}(0))$. Thus, it is only the horizontal foliation $\mathfrak{F}$ that really matters when the common time-dependent superposition rule is concerned. It is in a sense obvious, as composing Ψ with a diffeomorphism on $\mathbb{R}^n$ changes the superposition function (rearranges the level sets) but yields the same superposition rule. This proves the following (cf. [7]).

Proposition 6. *Giving a common time-dependent superposition rule (3) for a Lie family (1) is equivalent to giving a foliation which is horizontal with respect to the projection* $\pi : \mathbb{R} \times \mathbb{R}^{(m+1)n} \to \mathbb{R} \times \mathbb{R}^{nm}$ *and such that the vector fields* $\{\widetilde{Y}_\alpha\}_{\alpha \in \Lambda}$ *are tangent to their leaves.*

3 Generalized Lie Theorem.

It is generally difficult to determine whether a family (8) admits a common time-dependent superposition rule by means of Proposition 6. It is therefore interesting to find a characterization of Lie families by means of a more convenient criterion, e.g. through an easily verifiable condition based on the properties of the time-dependent vector fields $\{Y_\alpha\}_{\alpha \in \Lambda}$. Finding such a criterion is the main result of this section. It is formulated as Generalized Lie Theorem.

We start with three lemmata. The proofs of first two of them are straightforward.

Lemma 7. *Given two time-dependent vector fields* X *and* Y *on* $\mathbb{R}^n$*, the commutator* $[\widetilde{X}, \widetilde{Y}]$ *on* $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ *is the prolongation of a time-dependent vector field* Z *on* $\mathbb{R}^n$*,* $[\widetilde{X}, \widetilde{Y}] = \widehat{Z}$*.*

Lemma 8. *Given a family of time-dependent vector fields,* $X_1, \ldots, X_r$*, on* $\mathbb{R}^n$*, their autonomizations satisfy the relations*

$$[\bar{X}_j, \bar{X}_k](t, x) = \sum_{l=1}^{r} f_{jkl}(t) \bar{X}_l(t, x), \qquad j, k = 1, \ldots, r,$$

for some time-dependent functions $f_{jkl}:\mathbb{R}\to\mathbb{R}$, if and only if their time-prolongations to $\mathbb{R}\times\mathbb{R}^{n(m+1)}$, $\widetilde{X}_1,\ldots,\widetilde{X}_r$, satisfy analogous relations

$$[\widetilde{X}_j,\widetilde{X}_k](t,x)=\sum_{l=1}^r f_{jkl}(t)\widetilde{X}_l(t,x),\qquad j,k=1,\ldots,r.$$

Moreover, $\sum_{l=1}^r f_{jkl}(t)=0$ for all $j,k=1,\ldots,r$.

Lemma 9. *Consider a family of time-dependent vector fields, $Y_1,\ldots,Y_r$, with time prolongations to $\mathbb{R}\times\mathbb{R}^{n(m+1)}$, $\widetilde{Y}_1,\ldots,\widetilde{Y}_r$, such that their projections $\pi_*(\widetilde{Y}_j)$ are linearly independent at a generic point in $\mathbb{R}\times\mathbb{R}^{nm}$. Then, $\sum_{j=1}^r b_j\widetilde{Y}_j$, with $b_j\in C^\infty(\mathbb{R}\times\mathbb{R}^{nm})$, is of the form $\widehat{Y}$ (resp. $\widetilde{Y}$) for a time-dependent vector field Y on $\mathbb{R}^n$, if and only if the functions b_j depend on the time only, that is, $b_j=b_j(t)$, and $\sum_{j=1}^r b_j=0$ (resp., $\sum_{j=1}^r b_j=1$).*

Proof. We shall only detail the proof of the above claim for $\sum_{j=1}^r b_j\widetilde{Y}_j=\widehat{Y}$, as the proof of the other case is completely analogous. Let us write in coordinates

$$\widetilde{Y}_j=\frac{\partial}{\partial t}+\sum_{i=1}^n\sum_{a=0}^m A^i_j(t,x_{(a)})\frac{\partial}{\partial x^i_{(a)}},\qquad j=1,\ldots,r.$$

Then,

$$\sum_{j=1}^r b_j(t,x_{(0)},\ldots,x_{(m)})\widetilde{Y}_j=\sum_{j=1}^r\sum_{i=1}^n\sum_{a=0}^m b_j(t,x_{(0)},\ldots,x_{(m)})A^i_j(t,x_{(a)})\frac{\partial}{\partial x^i_{(a)}}+\sum_{j=1}^r b_j(t,x_{(0)},\ldots,x_{(m)})\frac{\partial}{\partial t},$$

which is a prolongation if and only if there are functions $B^i:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, with $i=1,\ldots,n$, such that

$$\left\{\begin{array}{l}\displaystyle\sum_{j=1}^r b_j(t,x_{(0)},\ldots,x_{(m)})A^i_j(t,x_{(a)})=B^i(t,x_{(a)}),\\ \displaystyle\sum_{j=1}^r b_j(t,x_{(0)},\ldots,x_{(m)})=0,\end{array}\right.\qquad a=0,\ldots,m,\quad i=1,\ldots,n.$$

If the functions $b_1,\ldots,b_r$ are time-dependent only and $\sum_{j=1}^r b_j=0$, the above conditions hold and $\sum_{j=1}^r b_j\widetilde{Y}_j$ is the prolongation to $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ of the time-dependent vector field $Y=\sum_{i=1}^n B^i(t,x)\partial/\partial x^i$.

Conversely, suppose that $\sum_{j=1}^r b_j\widetilde{Y}_j$ is a prolongation for a time-dependent vector field on $\mathbb{R}^n$. In this case, the functions $b_j(t,x_{(0)},\ldots,x_{(m)})$ solve the following system of linear

equations in the unknown variables u_α:

$$\begin{cases} \displaystyle\sum_{j=1}^{r} u_j A^i_j(t, x_{(a)}) = B^i(t, x_{(a)}), \\ \displaystyle\sum_{j=1}^{r} u_j = 0, \end{cases}$$

where $a = 1, \ldots, m$ and $i = 1, \ldots, n$. This is a system of $m \cdot n + 1$ equations and, as $\pi_*(\widetilde{Y}_j)$, with $j = 1, \ldots, r$, are linearly independent by assumption, the solutions u_α are uniquely determined by the variables $\{t, x_{(1)}, \ldots, x_{(m)}\}$ and therefore they do not depend on $x_{(0)}$. Since time-prolongations are invariant with respect to the symmetry group S_{m+1} acting on $\mathbb{R}^{n(m+1)} = (\mathbb{R}^n)^{m+1}$ in the obvious way, the functions $b_j(t, x_{(0)}, \ldots, x_{(m)})$, with $j = 1, \ldots, r$, must satisfy such a symmetry. Hence, as they do not depend on $x_{(0)}$, they cannot depend on the variables $\{x_{(1)}, \ldots, x_{(m)}\}$ and they are functions depending on the time only. $\square$

Theorem 10. (Generalized Lie Theorem) *The family of systems (1) admits a common time-dependent superposition rule if and only if the vector fields* $\{\bar{Y}_\alpha\}_{\alpha\in\Lambda}$ *can be written in the form*

$$\bar{Y}_\alpha(t, x) = \sum_{j=1}^{r} b_{\alpha j}(t)\bar{X}_j(t, x), \qquad \alpha \in \Lambda, \tag{6}$$

where $b_{\alpha j}$ *are functions of the time only,* $\sum_{j=1}^{n} b_{\alpha j} = 1$, *and,* $X_1, \ldots, X_r$, *are time-dependent vector fields such that*

$$[\bar{X}_j, \bar{X}_k](t, x) = \sum_{l=1}^{r} f_{jkl}(t)\bar{X}_l(t, x), \qquad j, k = 1, \ldots, r. \tag{7}$$

for some functions $f_{jkl} : \mathbb{R} \to \mathbb{R}$, *with* $j, k, l = 1, \ldots, r$. *We call the family of autonomizations,* $\bar{X}_1, \ldots, \bar{X}_r$, *a system of generators of the Lie family.*

Proof. Suppose first that the family of systems (1) admits a *common time-dependent superposition rule* and let $\mathfrak{F}$ be the corresponding n-codimensional horizontal foliation. The vector fields $\{\widetilde{Y}_\alpha\}_{\alpha\in\Lambda}$ are tangent to the leaves of the foliation $\mathfrak{F}$ and span a distribution $\mathcal{D}_0$ on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$. Such a distribution need not be involutive, see examples in Section 5. Nevertheless, we can enlarge the family $\{\widetilde{Y}_\alpha\}_{\alpha\in\Lambda}$ to the Lie algebra of vector fields generated by such a family. This Lie algebra is spanned by $\{\widetilde{Y}_\alpha\}_{\alpha\in\Lambda}$ and all their possible Lie brackets, i.e.,

$$\widetilde{Y}_\alpha,\ [\widetilde{Y}_\alpha, \widetilde{Y}_\beta],\ [\widetilde{Y}_\alpha, [\widetilde{Y}_\beta, \widetilde{Y}_\gamma]],\ [\widetilde{Y}_\alpha, [\widetilde{Y}_\beta, [\widetilde{Y}_\gamma, \widetilde{Y}_\delta]]], \ldots \qquad \alpha, \beta, \gamma, \delta, \ldots \in \Lambda. \tag{8}$$

All the above vector fields are tangent to the leaves of the foliation $\mathfrak{F}$ and therefore there are up to $m{\cdot}n{+}1$ linearly independent ones at a generic point of $\mathbb{R}{\times}\mathbb{R}^{n(m+1)}$. Consequently, they span an involutive generalized distribution $\mathcal{D}$ with leaves of dimension $r \le m \cdot n + 1$.

In a neighborhood of a regular point of this foliation, take now a finite basis of vector fields from the elements of the family (8) spanning the distribution. By construction, at least one of them must be of the form $\widetilde{X}_1$ for a certain time-dependent vector field X_1 on $\mathbb{R}^n$ and, in view of lemma 9 and the form of the family (8), those not being time-prolongations are just prolongations. Therefore, if we add $\widetilde{X}_1$ to those elements of the basis being prolongations, we get a new basis of the distribution $\mathcal{D}$ made up by certain $r \leq m \cdot n + 1$ time-prolongations $\widetilde{X}_1, \ldots, \widetilde{X}_r$. In other words, the distribution $\mathcal{D}$ is locally spanned, near regular points, by time-prolongations, say $\widetilde{X}_1, \ldots, \widetilde{X}_r$. As the generalized distribution $\mathcal{D}$ is involutive, there exist r^3 real functions f_{jkl}, with $j, k, l = 1, \ldots, r$, on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ such that

$$[\widetilde{X}_j, \widetilde{X}_k] = \sum_{l=1}^{r} f_{jkl} \widetilde{X}_l, \qquad j, k = 1, \ldots, r,$$

and as the left side of the above equalities are prolongations, we get that, in view of lemma 9, all the functions f_{jkl} depend on time only and $\sum_{l=1}^{n} f_{jkl} = 0$. Finally, taking into account Lemma 8, we have

$$[\bar{X}_j, \bar{X}_k](t, x) = \sum_{l=1}^{r} f_{jkl}(t) \bar{X}_l(t, x), \qquad j, k = 1, \ldots, r.$$

Note that as the vector fields $\{\widetilde{Y}_\alpha\}_{\alpha \in \Lambda}$ are contained in the distribution $\mathcal{D}$, there exist some functions $b_{\alpha j} \in C^\infty(\mathbb{R}^{n(m+1)})$ such that $\widetilde{Y}_\alpha = \sum_{j=1}^{r} b_{\alpha j} \widetilde{X}_j$ for every $\alpha \in \Lambda$. In consequence, according again to lemma 9, the functions $b_{\alpha j}$ depend on the time only, i.e., $b_{\alpha j} = b_{\alpha j}(t)$. Therefore, we get that

$$\widetilde{Y}_\alpha = \sum_{j=1}^{r} b_{\alpha j} \widetilde{X}_j \Longrightarrow \bar{Y}_\alpha(t, x) = \sum_{j=1}^{r} b_{\alpha j}(t) \bar{X}_j(t, x), \qquad \alpha \in \Lambda.$$

Let us prove the converse. Assume that we can write

$$\bar{Y}_\alpha(t, x) = \sum_{j=1}^{r} b_{\alpha j}(t) \bar{X}_j(t, x)$$

for certain time-dependent vector fields $X_1, \ldots, X_r$ on $\mathbb{R}^n$ such that

$$[\bar{X}_j, \bar{X}_k](t, x) = \sum_{l=1}^{r} f_{jkl}(t) \bar{X}_l(t, x), \qquad j, k = 1, \ldots, r.$$

In view of lemma 8, the vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_r$ span an involutive distribution $\mathcal{D}$ on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ for any m. Furthermore, the rank of this distribution is not greater than r and therefore, for m big enough, this distribution is at least n-codimensional and it gives rise to a foliation $\mathfrak{F}_0$ which is horizontal with respect to the projection π. Moreover, if codimension of $\mathfrak{F}_0$ is bigger than n, we can enlarge $\mathfrak{F}_0$ to a n-codimensional foliation $\mathfrak{F}$, still horizontal with respect to the map π giving rise to a common time-dependent superposition rule for the family (1). $\square$

4 Lie families, quasi-Lie and Lie systems

This section is devoted to recalling the theories of quasi-Lie schemes and Lie systems needed to investigate the relations among these theories and Lie families. A full detailed report on these topics can be found in [7, 17].

The theory of quasi-Lie schemes provides various results on the transformation properties of time-dependent vector fields by a certain kind of time-dependent changes of variables associated with generalized flows.

Each time-dependent vector field X gives rise to a *generalized flow* g^X, i.e., a map $g^X : (t,x) \in \mathbb{R} \times \mathbb{R}^n \to g^X_t(x) \equiv g^X(t,x) \in \mathbb{R}^n$ (more precisely, defined in a neighborhood of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$), with $g^X_0 = \mathrm{Id}_{\mathbb{R}^n}$, such that the curve $\gamma^X_{x_0}(t) = g^X_t(x_0)$ is the integral curve of the time-dependent vector field X starting from the point $x_0 \in \mathbb{R}^n$, i.e., $\dot{\gamma}^X_{x_0} = X(t, \gamma^X_{x_0}(t))$ and $\gamma^X_{x_0}(0) = g^X_0(x_0) = x_0$.

Denote with $\mathfrak{X}_t(\mathbb{R}^n)$ the set of all time-dependent vector fields on $\mathbb{R}^n$. Each generalized flow h acts on the set of time-dependent vector fields $\mathfrak{X}_t(\mathbb{R}^n)$ transforming each time-dependent vector field $X \in \mathfrak{X}_t(\mathbb{R}^n)$ into a new one, $h_\bigstar X$, with the generalized flow of the form $g^{h_\bigstar X} = h \circ g^X$. In terms of autonomizations we can write ([17, Theorem 3])

$$\overline{h_\bigstar X} = \bar{h}_* \bar{X}\,,$$

where $\bar{h}$ is the natural *autonomization* of the generalized flow h to a (local) diffeomorphism of $\mathbb{R} \times \mathbb{R}^n$, $\bar{h}(t,x) = (t, h_t(x))$ and where $\bar{h}_*$ is the standard action of the diffeomorphism $\bar{h}$ on vector fields. This, in turn, implies that

$$[\overline{h_\bigstar X}, \overline{h_\bigstar Y}] = \bar{h}_*[\bar{X}, \bar{Y}]\,. \tag{9}$$

Let V be a finite-dimensional vector space of vector fields on $\mathbb{R}^n$. We denote with $V(\mathbb{R})$ the set of time-dependent vector fields $X \in \mathfrak{X}_t(\mathbb{R}^n)$ with values in V, that is, those time-dependent vector fields X such that, for every $t \in \mathbb{R}$, the vector field $X_t(x)$ belongs to V. In terms of the introduced terminology and notation, Lie Theorem, whose statement can be found for instance in [1, 7], can be reformulated as follows.

Proposition 11. (Lie Theorem) *A non-autonomous system X is a Lie system on $\mathbb{R}^n$ if and only if there exists a finite-dimensional Lie algebra of vector fields $V_0 \subset \mathfrak{X}(\mathbb{R}^n)$ such that $X \in V_0(\mathbb{R})$.*

Definition 12. A *quasi-Lie scheme* $S(W,V)$ on the manifold M consists of two finite-dimensional vector spaces of vector fields $W, V \subset \mathfrak{X}(M)$ such that

- W is a linear subspace of V.
- W is a Lie algebra of vector fields, that is, $[W,W] \subset W$.
- W normalizes V, i.e., $[W,V] \subset V$.

It has been proved in [17] that given a quasi-Lie scheme $S(W,V)$, the space $V(\mathbb{R})$ is stable under the action of the infinite-dimensional group $\mathcal{G}(W)$ of generalized flows of vector fields in $W(\mathbb{R})$, i.e., $g_\bigstar X \in V(\mathbb{R})$, for every $X \in V(\mathbb{R})$ and $g \in \mathcal{G}(W)$.

Definition 13. Given a quasi-Lie scheme $S(W,V)$, we say that a time-dependent vector field $X \in V(\mathbb{R})$ is a *quasi-Lie system* with respect to this scheme, if there exist a generalized flow $g \in \mathcal{G}(W)$ and a Lie algebra of vector fields $V_0 \subset V$, such that $g_\bigstar X \in V_0(\mathbb{R})$.

As for each Lie system X there exists a Lie algebra of vector fields V_0 such that $X \in V_0(\mathbb{R})$, it is obvious that $X \in S(V_0, V_0)$ and, consequently, every Lie system is also a quasi-Lie system.

From now on, given a quasi-Lie scheme $S(W,V)$, a generalized flow $g \in \mathcal{G}(W)$, and a Lie algebra of vector fields $V_0 \subset V$, we denote with $S_g(W,V;V_0)$ the set of quasi-Lie systems of the scheme $S(W,V)$ such that $g_\bigstar X \in V_0(\mathbb{R})$.

Proposition 14. *The family of quasi-Lie systems $S_g(W,V;V_0)$ is a Lie family admitting the common time-dependent superposition function of the form*

$$\bar{\Phi}_g(t, x_{(1)}, \ldots, x_{(m)}, k) = g_t^{-1} \circ \Phi\left(g_t(x_{(1)}), \ldots, g_t(x_{(m)}), k\right) , \tag{10}$$

for any time-independent superposition function Φ associated with the Lie algebra of vector fields V_0 by Lie Theorem.

Proof. Let $Z_1, \ldots, Z_r$ be a basis in V_0. Since V_0 is closed with respect to the Lie bracket,

$$[Z_j, Z_k] = \sum_{l=0}^{r} c_{jkl} Z_l \tag{11}$$

for some constants c_{jkl}, $j,k,l = 1, \ldots, r$. For any $Y \in S_g(W,V;V_0)$, there exist functions b_j such that

$$(g_\bigstar Y)_t(x) = \sum_{j=1}^{r} b_j(t) Z_j(x)\,.$$

Consequently, for the autonomization we can write

$$\overline{g_\bigstar Y}(t,x) = \sum_{j=0}^{r} b_j(t) \bar{Z}_j(t,x)\,, \tag{12}$$

where we put $Z_0 = 0$ (thus $\bar{Z}_0 = \partial/\partial t$) and $b_0(t) = 1 - \sum_{j=1}^{r} b_j(t)$.

Note that, as Z_k are time-independent, the autonomizations $\bar{Z}_k$, $k = 0, \ldots, r$, form a Lie algebra:

$$[\bar{Z}_j, \bar{Z}_k] = \sum_{l=0}^{r} c_{jkl} \bar{Z}_l\,, \tag{13}$$

where $c_{0kl} = c_{j0l} = 0$ and $c_{jk0} = -\sum_{l=1}^{r} c_{jkl}$, for $j,k = 1, \ldots, r$. Hence, according to (9), the autonomizations $\bar{Z}'_k = \overline{g_\bigstar^{-1}(Z_k)}$ are also closed with respect to the bracket,

$$[\bar{Z}'_j, \bar{Z}'_k] = \sum_{l=0}^{r} c_{jkl} \bar{Z}'_l\,, \tag{14}$$

and, in view of (12), the autonomization of any $Y \in S_g(W, V; V_0)$ can be written in the form

$$\bar{Y}(t,x) = \sum_{j=0}^{r} b_j(t)\bar{Z}'_j(t,x)\,.$$

This means, in view of Theorem 10, that $S_g(W, V; V_0)$ is a Lie family. The form (10) can be now easily derived (see [17, Theorem 4]). □

In view of the above proposition, every quasi-Lie system and, consequently, every Lie system can be included in a Lie family satisfying Theorem 10. This fact justifies once more calling this theorem Generalized Lie Theorem.

5 Applications.

In this section we will apply common time-dependent superposition rules for studying some first- and second-order differential equations. In this way, we will show how that common time-dependent superposition rules can be used to analyze equations which cannot be studied by means of the usual theory of Lie systems. Additionally, some new results for the study of Abel and Milne–Pinney equations are provided.

5.1 A time-dependent superposition rule for Abel equations

We illustrate here our theory by deriving a common time-dependent superposition rule for a Lie family of Abel equations whose elements do not admit a standard superposition rule except for a few particular instances. In this way, we single out that our theory provides new tools for investigating solutions of non-autonomous systems of differential equations than cannot be analyzed by means of the theory of Lie systems.

With this aim, we analyze the so-called Abel equations of the first-type [20, 21], i.e., the differential equations of the form

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3, \tag{15}$$

with $a_3(t) \neq 0$. Abel equations appear in the analysis of several cosmological models [22, 23, 24] and other different fields in Physics [25, 26, 27, 28, 29, 30]. Additionally, the study of integrability conditions for Abel equations is a research topic of current interest in Mathematics and multiple studies have been carried out in order to analyze the properties of the solutions of these equations [21, 31, 32, 33, 34].

Note that, apart from its inherent mathematical interest, the knowledge of particular solutions of Abel equations allows us to study the properties of those physical systems that such equations describe. Thus, the expressions enabling us to obtain easily new solutions of Abel equations by means of several particular ones, like common time-dependent superposition rules, are interesting to study the solutions of these equations and, therefore, their related physical systems.

Unfortunately, all the expressions describing the general solution of Abel equations presently known can only be applied to study autonomous instances and, moreover, they depend on families of particular conditions satisfying certain extra conditions, see [31, 32]. Taking this into account, common time-dependent superposition rules represent an improvement with respect to these previous expressions, as they permit one to treat non-autonomous Abel equations and they do not require the use of particular solutions satisfying additional conditions.

Recall that, according to Theorem 10, the existence of a common time-dependent superposition rule for a family of time-dependent vector fields (2) requires the existence of a system of generators, i.e., a certain set of time-dependent vector fields, $X_1, \ldots, X_r$, satisfying relations (7). Conversely, given such a set, the family of time-dependent vector fields Y whose autonomizations can be written in the form

$$\bar{Y}_\alpha(t,x) = \sum_{j=1}^{r} b_j(t)\bar{X}_j(t,x), \qquad \sum_{j=1}^{r} b_j(t) = 1,$$

admits a common time-dependent superposition rule and becomes a Lie family.

Consequently, a Lie family of Abel equations can be determined, for instance, by finding two time-dependent vector fields of the form

$$\begin{aligned} X_1(t,x) &= (b_0(t) + b_1(t)x + b_2(t)x^2 + b_3(t)x^3)\frac{\partial}{\partial x}, \\ X_2(t,x) &= (b'_0(t) + b'_1(t)x + b'_2(t)x^2 + b'_3(t)x^3)\frac{\partial}{\partial x}, \qquad b'_3(t) \neq 0, \end{aligned} \tag{16}$$

such that

$$[\bar{X}_1, \bar{X}_2] = 2(\bar{X}_2 - \bar{X}_1). \tag{17}$$

Let us analyze the existence of such two time-dependent vector fields X_1 and X_2 holding relation (17). In coordinates, the Lie bracket $[\bar{X}_1, \bar{X}_2]$ reads

$$\begin{aligned} [(b'_3b_2 - b'_2b_3)x^4 + (2(b'_3b_1 - b_3b'_1) - \dot{b}_3 + \dot{b}'_3)x^3 + (-3(b'_0b_3 - b_0b'_3) + (b'_2b_1 - b_2b'_1) \\ - \dot{b}_2 + \dot{b}'_2)x^2 + (-2b'_0b_2 + 2b_0b'_2 - \dot{b}_1 + \dot{b}'_1)x - b'_0b_1 + b_0b'_1 - \dot{b}_0 + \dot{b}'_0]\frac{\partial}{\partial x}. \end{aligned}$$

Hence, in order to satisfy condition (17), $b'_3b_2 - b'_2b_3 = 0$, e.g. we may fix $b_2 = b_3 = 0$. Additionally, for the sake of simplicity, we assume $b'_3 = 1$. In this case, the previous expression takes the form

$$[2b_1x^3 + (3b_0 + b'_2b_1 + \dot{b}'_2)x^2 + (2b_0b'_2 - \dot{b}_1 + \dot{b}'_1)x - b'_0b_1 + b_0b'_1 - \dot{b}_0 + \dot{b}'_0]\frac{\partial}{\partial x},$$

and, taking into account the values chosen for b_2, b_3 and b'_3, assumption (17) yields $b_1 = 1$ and

$$\begin{cases} b'_2 = 3b_0 + \dot{b}'_2, \\ 2(b'_1 - 1) = 2b_0b'_2 + \dot{b}'_1, \\ 2(b'_0 - b_0) = -b'_0 + b_0b'_1 - \dot{b}_0 + \dot{b}'_0. \end{cases}$$

As the above system has more variables than equations, we can try to fix some values of the variables in order to simplify it and obtain a particular solution. In this way, taking $b_0(t)=t$, the above system reads

$$\begin{cases} \dot{b}'_2 = b'_2 - 3t, \\ \dot{b}'_1 = 2(b'_1-1) - 2tb'_2, \\ \dot{b}'_0 = 2(b'_0 - t) + b'_0 - tb'_1 + 1. \end{cases}$$

This system is integrable by quadratures and it can be verified that it admits the particular solution

$$b'_2(t) = 3(1+t), \quad b'_1(t) = 3(1+t)^2+1, \quad b'_0(t) = (1+t)^3 + t.$$

Summing up, we have proved that the time-dependent vector fields

$$\begin{cases} X_1(t,x) = (t+x)\dfrac{\partial}{\partial x}, \\ X_2(t,x) = ((1+t)^3 + t + (3(1+t)^2+1)x + 3(1+t)x^2 + x^3)\dfrac{\partial}{\partial x}, \end{cases} \tag{18}$$

satisfy (17) and, therefore, the family of time-dependent vector fields $Y_{b(t)}(t,x) = (1-b(t))X_1(x) + b(t)X_2(x)$ is a Lie family. The corresponding family of Abel equations is

$$\frac{dx}{dt} = (t+x) + b(t)(1+t+x)^3. \tag{19}$$

According to the results proved in Section 3, in order to determine a common time-dependent superposition rule for the above Lie family we have to determine a first-integral for the vector fields of the distribution $\mathcal{D}$ spanned by the time-prolongations $\widetilde{X}_1$ and $\widetilde{X}_2$ on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ for a certain m so that the time-prolongations of X_1 and X_2 to $\mathbb{R}\times\mathbb{R}^{nm}$ were linearly independent at a generic point. Taking into account expressions (18), the prolongations of the vector fields X_1 and X_2 to $\mathbb{R}\times\mathbb{R}^2$ are linearly independent at a generic point and, in view of (17), the time-prolongations $\widetilde{X}_1$ and $\widetilde{X}_2$ to $\mathbb{R}\times\mathbb{R}^3$ span an involutive generalized distribution $\mathcal{D}$ with leaves of dimension two in a dense subset of $\mathbb{R}\times\mathbb{R}^3$. Finally, a first-integral for the vector fields in the distribution $\mathcal{D}$ will provide us a common time-dependent superposition rule for the Lie family (19).

Since, in view of (17), the vector fields $\widetilde{X}_1$ and $\widetilde{X}_2$ span the distribution $\mathcal{D}$, a function $G:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ is a first-integral of the vector fields of the distribution $\mathcal{D}$ if and only if G is a first-integral of $\widetilde{X}_1$ and $\widetilde{X}_1-\widetilde{X}_2$, i.e. $\widetilde{X}_1G = (\widetilde{X}_2-\widetilde{X}_1)G = 0$.

The condition $\widetilde{X}_1G=0$ reads

$$\frac{\partial G}{\partial t} + (t+x_0)\frac{\partial G}{\partial x_0} + (t+x_1)\frac{\partial G}{\partial x_1} = 0,$$

and, using the method of characteristics [35], we note that the curves on which G is constant, the so-called *characteristics*, are solutions of the system

$$dt = \frac{dx_0}{t+x_0} = \frac{dx_1}{t+x_1} \Rightarrow \frac{dx_i}{dt} = t + x_i, \qquad i=0,1,$$

i.e., $x_i(t) = \xi_i e^t - t - 1$, with $i = 0, 1$. These solutions are determined by the implicit equations $\xi_0 = e^{-t}(x_0 + t + 1)$ and $\xi_1 = e^{-t}(x_1 + t + 1)$, with $\xi_0, \xi_1 \in \mathbb{R}$. Therefore, there exists a function $G_2 : \mathbb{R}^2 \to \mathbb{R}$ such that $G(t, x_0, x_1) = G_2(\xi_0, \xi_1)$. In other words, each first-integral G of $\widetilde{X}_1$ depends only on ξ_0 and ξ_1.

Taking into account the previous fact, we look for first-integrals of the vector field $\widetilde{X}_2 - \widetilde{X}_1$ being also first-integrals of $\widetilde{X}_1$, that is, for solutions of the equation $(\widetilde{X}_2 - \widetilde{X}_1)G = 0$ with G depending on ξ_0 and ξ_1. Using the expression of $\widetilde{X}_2 - \widetilde{X}_1$ in the system of coordinates $\{t, \xi_0, \xi_1\}$, we get that

$$\xi_0^3 \frac{\partial G}{\partial \xi_0} + \xi_1^3 \frac{\partial G}{\partial \xi_1} = \xi_0^3 \frac{\partial G_2}{\partial \xi_0} + \xi_1^3 \frac{\partial G_2}{\partial \xi_1} = 0,$$

and, applying again the method of characteristics, we obtain that there exists a function $G_3 : \mathbb{R} \to \mathbb{R}$ such that $G(t, x_0, x_1) = G_2(\xi_0, \xi_1) = G_3(\Delta)$, where $\Delta = e^{2t}((x_0 + t + 1)^{-2} - (x_1 + t + 1)^{-2})$. Finally, using this first-integral, we get that the common time-dependent superposition rule for the Lie family (19) reads

$$k = e^{2t}((x_0 + t + 1)^{-2} - (x_1 + t + 1)^{-2}),$$

with k being a real constant. Therefore, given any particular solution $x_1(t)$ of a particular instance of the family of first-order Abel equations (21), the general solution, $x(t)$, of this instance is

$$x(t) = \left((x_1(t) + t + 1)^{-2} + ke^{-2t}\right)^{-1/2} - t - 1.$$

Note that our previous procedure can be straightforwardly generalized to derive common time-dependent superposition rules for generalized Abel equations [36], i.e., the differential equations of the form

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 + \ldots + a_n(t)x^n, \qquad n \geq 3.$$

Actually, their study can be approached by analyzing the existence of two vector fields of the form

$$\begin{aligned} Y_1(t, x) &= (b_0(t) + b_1(t)x + \ldots + b_n(t)x^n)\frac{\partial}{\partial x}, \\ Y_2(t, x) &= (b'_0(t) + b'_1(t)x + \ldots + b'_n(t)x^n)\frac{\partial}{\partial x}, \qquad b'_n(t) \neq 0, \end{aligned}$$

satisfying the relation $[\bar{Y}_1, \bar{Y}_2] = 2(\bar{Y}_2 - \bar{Y}_1)$ and following a procedure similar to the one developed above.

5.2 Lie families and second-order differential equations

Common time-dependent superposition rules describe solutions of non-autonomous systems of first-order differential equations. Nevertheless, we shall now illustrate how this new kind of superposition rules can be applied to analyze also families of second-order

differential equations. More specifically, we shall derive a common time-dependent superposition rule in order to express the general solution of any instance of a family of Milne–Pinney equations [31, 37, 38] in terms of each generic pair of particular solutions, two constants, and the time. In this way, we provide a generalization to the setting of dissipative Milne–Pinney equations of the expression previously derived to analyze the solutions of Milne–Pinney equations in [11].

Consider the family of dissipative Milne–Pinney equations [37, 38, 39, 40] of the form

$$\ddot{x} = -\dot{F}\dot{x} + \omega^2 x + e^{-2F} x^{-3}, \tag{20}$$

with a fixed time-dependent function $F = F(t)$, and parametrized by an arbitrary time-dependent function $\omega = \omega(t)$. The physical motivation for the study of dissipative Milne–Pinney equations comes from its appearance in dissipative quantum mechanics [41, 42, 43, 44], where, for instance, their solutions are used to obtain Gaussian solutions of non-conservative time-dependent quantum oscillators [43]. Moreover, the mathematical properties of the solutions of dissipative Milne–Pinney equations have been studied by several authors from different points of view as well as for different purposes [11, 17, 18, 37, 38, 45, 46, 47]. As relevant instances, consider the works [18, 37] which outline the state-of-the-art of the investigation of dissipative and non-dissipative Milne–Pinney equations. One of the main achievements on this topic (see [37, Corollary 5]) is concerned with an expression describing the general solution of a particular class of these equations in terms of a pair of generic particular solutions of a second-order linear differential equations and two constants. Recently the theory of quasi-Lie schemes and the theory of Lie systems enabled us to recover this latter result and other new ones from a geometric point of view [10, 17].

Note that introducing a new variable $v \equiv \dot{x}$, we transform the family (20) of second-order differential equations into a family of first-order ones

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\dot{F}v + \omega^2 x + e^{-2F} x^{-3}, \end{cases} \tag{21}$$

whose dynamics is described by the following family of time-dependent vector fields on $\mathrm{T}\mathbb{R}$ parametrized by ω,

$$Y_\omega = \left(-\dot{F}v + e^{-2F}x^{-3} + \omega^2 x\right)\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}, \qquad \omega \in \Lambda = C^\infty(t).$$

Let us show that the above family is a Lie family whose common superposition rule can be used to analyze the solutions of the family (20).

In view of Theorem 10, if the family of systems related to the above family of time-dependent vector fields is a Lie family, that is, it admits a common time-dependent superposition rule in terms of m particular solutions, then the family of vector fields on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ given by

$$\widetilde{Y}_\omega,\ [\widetilde{Y}_\omega, \widetilde{Y}_{\omega'}],\ [\widetilde{Y}_\omega, [\widetilde{Y}_{\omega'}, \widetilde{Y}_{\omega''}]],\ [\widetilde{Y}_\omega, [\widetilde{Y}_{\omega'}, [\widetilde{Y}_{\omega''}, \widetilde{Y}_{\omega'''}]]] \ldots,\ \omega, \omega', \omega'', \omega''', \ldots \in \Lambda, \tag{22}$$

spans an involutive generalized distribution with leaves of rank $r \leq n \cdot m + 1$.

Note that the distribution spanned by all $\widetilde{Y}_\omega$ is generated by the vector fields $\widetilde{Y}_1$ and $\widetilde{Y}_2$, with

$$Y_1 = \left(-\dot{F}v + e^{-2F}x^{-3} + x\right)\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}, \quad Y_2 = \left(-\dot{F}v + e^{-2F}x^{-3}\right)\frac{\partial}{\partial v} + v\frac{\partial}{\partial x},$$

since $\widetilde{Y}_\omega = (1-\omega^2)\widetilde{Y}_2 + \omega^2\widetilde{Y}_1$. It is easy to see that the prolongation $[\widetilde{Y}_1, \widetilde{Y}_2]$ is not spanned by $\widetilde{Y}_1$ and $\widetilde{Y}_2$ and, so that we have to include the prolongation $\widehat{Y}_3 = [\widetilde{Y}_1, \widetilde{Y}_2]$ to the picture, where

$$Y_3 = x\frac{\partial}{\partial x} - (v + x\dot{F})\frac{\partial}{\partial v}.$$

In the case $m = 0$, the distribution spanned by the vector fields, $\widetilde{Y}_1, \widetilde{Y}_2, \widehat{Y}_3$, does not admit a non-trivial first-integral. In the case $m > 0$, the vector fields $\widetilde{Y}_1, \widetilde{Y}_2, \widehat{Y}_3$ do not span all the elements of family (22) and we need to add to them the prolongation $\widehat{Y}_4 = [\widetilde{Y}_1, [\widetilde{Y}_1, \widetilde{Y}_2]]$, with

$$Y_4 = (2v + x\dot{F})\frac{\partial}{\partial x} + (2e^{-2F}x^{-3} - 2x - \dot{F}(v + x\dot{F}) - x\ddot{F})\frac{\partial}{\partial v}.$$

The vector fields $\widetilde{Y}_1, \widetilde{Y}_2, \widehat{Y}_3, \widehat{Y}_4$ satisfy the commutation relations

$$\begin{aligned}
\left[\widetilde{Y}_1, \widetilde{Y}_2\right] &= \widehat{Y}_3, \\
\left[\widetilde{Y}_1, \widehat{Y}_3\right] &= \widehat{Y}_4, \\
\left[\widetilde{Y}_1, \widehat{Y}_4\right] &= (4 + \dot{F}^2 + 2\ddot{F})\widehat{Y}_3 - (\dot{F}\ddot{F} + \dddot{F})(\widetilde{Y}_1 - \widetilde{Y}_2), \\
\left[\widetilde{Y}_2, \widehat{Y}_3\right] &= 2(\widetilde{Y}_1 - \widetilde{Y}_2) + \widehat{Y}_4, \\
\left[\widetilde{Y}_2, \widehat{Y}_4\right] &= (2 + \dot{F}^2 + 2\ddot{F})\widehat{Y}_3 - (\dot{F}\ddot{F} + \dddot{F})(\widetilde{Y}_1 - \widetilde{Y}_2), \\
\left[\widehat{Y}_3, \widehat{Y}_4\right] &= -2\widehat{Y}_4 - 2(\widetilde{Y}_1 - \widetilde{Y}_2)(4 + \dot{F}^2 + 2\ddot{F}).
\end{aligned}$$

Consequently, the vector fields $\widetilde{Y}_1, \widetilde{Y}_2, \widehat{Y}_3, \widehat{Y}_4$ span the vector fields of the family (22). Adding $\widetilde{Y}_1$ to each prolongation of the previous set, that is, considering the vector fields $\widetilde{X}_1 = \widetilde{Y}_1$, $\widetilde{X}_2 = \widetilde{Y}_2$, $\widetilde{X}_3 = \widetilde{Y}_1 + \widehat{Y}_3$, and $\widetilde{X}_4 = \widetilde{Y}_1 + \widehat{Y}_4$, we get that the family of time-prolongations, $\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3, \widetilde{X}_4$, which spans the vector fields of the family (22). The com-

mutation relations among them read

$$
\begin{aligned}
&\left[\widetilde{X}_1,\widetilde{X}_2\right]=\widetilde{X}_3-\widetilde{X}_1,\\
&\left[\widetilde{X}_1,\widetilde{X}_3\right]=\widetilde{X}_4-\widetilde{X}_1,\\
&\left[\widetilde{X}_1,\widetilde{X}_4\right]=-(\dot F\ddot F+\dddot F+4+\dot F^2+2\ddot F)\widetilde{X}_1+(\dot F\ddot F+\dddot F)\widetilde{X}_2+(4+\dot F^2+2\ddot F)\widetilde{X}_3,\\
&\left[\widetilde{X}_2,\widetilde{X}_3\right]=2\widetilde{X}_1-2\widetilde{X}_2-\widetilde{X}_3+\widetilde{X}_4,\\
&\left[\widetilde{X}_2,\widetilde{X}_4\right]=-(1+\dot F^2+2\ddot F+\dot F\ddot F+\dddot F)\widetilde{X}_1+(\dot F\ddot F+\dddot F)\widetilde{X}_2+(1+\dot F^2+2\ddot F)\widetilde{X}_3,\\
&\left[\widetilde{X}_3,\widetilde{X}_4\right]=-3\widetilde{X}_4+(4+\dot F^2+2\ddot F)\widetilde{X}_3+(8+\dddot F+\dot F\ddot F+2\dot F^2+4\ddot F)\widetilde{X}_2+\\
&\qquad+(-9-3\dot F^2-6\ddot F-\dot F\ddot F-\dddot F)\widetilde{X}_1.
\end{aligned}
$$

As a consequence of Lemma 9, we get that the vector fields $\bar X_1$, $\bar X_2$ $\bar X_3$ and $\bar X_4$ close on the same commutation relations as the vector fields $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$, $\widetilde{X}_4$. Hence, in view of Theorem 10, the family (21) is a Lie family and the knowledge of non-trivial first-integrals of the vector fields of the distribution $\mathcal{D}$ spanned by $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$, $\widetilde{X}_4$ provides us with a common time-dependent superposition rule.

Note that, as the vector fields $\widetilde{X}_1$, $\widetilde{X}_1-\widetilde{X}_2$ and their Lie brackets span the whole distribution $\mathcal{D}$, a function $G:\mathbb{R}\times\mathrm{T}\mathbb{R}^3\to\mathbb{R}$ is a first-integral for the vector fields of the distribution $\mathcal{D}$ if and only if it is a first-integral for the vector fields $\widetilde{X}_1$ and $\widetilde{X}_2-\widetilde{X}_1$. Therefore, we can reduce the problem of finding first-integrals for the vector fields of the distribution $\mathcal{D}$ to finding common first-integrals G for the vector fields $\widetilde{X}_1$ and $\widetilde{X}_1-\widetilde{X}_2$.

Let us analyze the implications of G being a first-integral of the vector field

$$\widetilde{X}_1-\widetilde{X}_2=\sum_{i=0}^{2}x_i\frac{\partial}{\partial v_i}.$$

The characteristics of the above vector field are the solutions of the system

$$\frac{dv_0}{x_0}=\frac{dv_1}{x_1}=\frac{dv_2}{x_2},\qquad dx_0=0,\quad dx_1=0,\quad dx_2=0,\quad dt=0,$$

that is, the solutions are curves in $\mathbb{R}\times\mathrm{T}\mathbb{R}^3$ of the form $s\mapsto(t,x_0,x_1,x_2,v_0(s),v_1(s),v_2(s))$, with $\xi_{02}=x_0v_2(s)-x_2v_0(s)$ and $\xi_{12}=x_1v_2(s)-x_2v_1(s)$ for two real constants ξ_{02} and ξ_{12}. Thus, there exists a function $G_2:\mathbb{R}^6\to\mathbb{R}$ such that $G(p)=G_2(t,x_0,x_1,x_2,\xi_{02},\xi_{12})$, with $p\in\mathbb{R}\times\mathrm{T}\mathbb{R}^3$, $\xi_{02}=x_0v_2-x_2v_0$, and $\xi_{12}=x_1v_2-v_1x_2$. In other words, G is a function of $t,x_0,x_1,x_2,\xi_{02},\xi_{12}$.

The function G also satisfies the condition $\widetilde{X}_1G=0$ which, in terms of the coordinate system $\{t,x_0,x_1,x_2,\xi_{02}\xi_{12},v_2\}$, reads

$$
\begin{aligned}
\widetilde{X}_1G=\frac{\partial G}{\partial t}&+\frac{(x_0v_2-\xi_{02})}{x_2}\frac{\partial G}{\partial x_0}+\frac{(x_1v_2-\xi_{12})}{x_2}\frac{\partial G}{\partial x_1}+v_2\frac{\partial G}{\partial x_2}-\\
&-\left[\dot F\xi_{12}+e^{-2F}\left(\frac{x_2}{x_1^3}-\frac{x_1}{x_2^3}\right)\right]\frac{\partial G}{\partial\xi_{12}}-\left[\dot F\xi_{02}+e^{-2F}\left(\frac{x_2}{x_0^3}-\frac{x_0}{x_2^3}\right)\right]\frac{\partial G}{\partial\xi_{02}}=0.
\end{aligned}
$$

That is, defining the vector fields

$$\Xi_1 = \frac{\partial}{\partial t} - \frac{\xi_{12}}{x_2}\frac{\partial}{\partial x_1} - \frac{\xi_{02}}{x_2}\frac{\partial}{\partial x_0} + \left[-\dot{F}\xi_{12} - e^{-2F}\left(\frac{x_2}{x_1^3} - \frac{x_1}{x_2^3}\right)\right]\frac{\partial}{\partial \xi_{12}}$$
$$+ \left[-\dot{F}\xi_{02} - e^{-2F}\left(\frac{x_2}{x_0^3} - \frac{x_0}{x_2^3}\right)\right]\frac{\partial}{\partial \xi_{02}},$$
$$\Xi_2 = \frac{x_0}{x_2}\frac{\partial}{\partial x_0} + \frac{x_1}{x_2}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},$$

the condition $\widetilde{X}_1 G = 0$ implies that $\Xi_1 G_2 + v_2 \Xi_2 G_2 = 0$ and, as G_2 does not depend on v_2, the function G must be simultaneously a first-integral for Ξ_1 and Ξ_2, i.e., $\Xi_1 G = 0$ and $\Xi_2 G = 0$.

Applying again the method of characteristics to the vector field Ξ_2, we get that F can depend just on the variables $t, \xi_{02}, \xi_{12}, \Delta_{02} = x_0/x_2$ and $\Delta_{12} = x_1/x_2$, that is, there exists a function $G_3 : \mathbb{R}^5 \to \mathbb{R}$ such that $G(t, x_0, x_1, x_2, v_0, v_1, v_2) = G_2(t, x_0, x_1, x_2, \xi_{02}, \xi_{12}) = G_3(t, \xi_{02}, \xi_{12}, \Delta_{02}, \Delta_{12})$.

We are left to check out the implications of the equation $\Xi_1 G = 0$. Using the coordinate system $\{t, \xi_{02}, \xi_{12}, \Delta_{02}, \Delta_{12}, v_2, x_2\}$ and taking into account that $G(t, x_0, x_1, x_2, v_0, v_1, v_2) = G_3(t, \xi_{02}, \xi_{12}, \Delta_{02}, \Delta_{12})$, the previous equation can be cast into the form $\Xi_1 G = \frac{1}{x_2^2}\Upsilon_1 G_3 + \Upsilon_2 G_3 = 0$, where

$$\Upsilon_1 = \sum_{i=0}^{1}\left(-\xi_{i2}\frac{\partial}{\partial \Delta_{i2}} - e^{-2F}\left(\Delta_{i2}^{-3} - \Delta_{i2}\right)\frac{\partial}{\partial \xi_{i2}}\right),$$
$$\Upsilon_2 = -\dot{F}\xi_{12}\frac{\partial}{\partial \xi_{12}} - \dot{F}\xi_{02}\frac{\partial}{\partial \xi_{02}} + \frac{\partial}{\partial t}.$$

As G_3 depends on the variables $t, \Delta_{02}, \Delta_{12}, \xi_{12}, \xi_{02}$ only, we have that $\Upsilon_1 G = 0$ and $\Upsilon_2 G = 0$. Repeating *mutatis mutandis* the previous procedures in order to determine the implications of being a first-integral of Υ_1 and Υ_2, we finally get that the first-integrals of the distribution $\mathcal{D}$ are functions of I_1, I_2 and I, with

$$I_i = e^{2F}(x_0 v_i - x_i v_0)^2 + \left[\left(\frac{x_0}{x_i}\right)^2 + \left(\frac{x_i}{x_0}\right)^2\right], \qquad i = 1, 2,$$

and

$$I = e^{2F}(x_1 v_2 - x_2 v_1)^2 + \left[\left(\frac{x_1}{x_2}\right)^2 + \left(\frac{x_2}{x_1}\right)^2\right].$$

Defining $\bar{v}_2 = e^F v_2, \bar{v}_1 = e^F v_1$ and $\bar{v}_0 = e^F v_0$, the above first-integrals read

$$I_i = (x_0 \bar{v}_i - x_i \bar{v}_0)^2 + \left[\left(\frac{x_0}{x_i}\right)^2 + \left(\frac{x_i}{x_0}\right)^2\right], \qquad i = 1, 2,$$

and

$$I = (x_1\bar{v}_2 - x_2\bar{v}_1)^2 + \left[\left(\frac{x_1}{x_2}\right)^2 + \left(\frac{x_2}{x_1}\right)^2\right].$$

Note that these first-integrals have the same form as the ones considered in [10] for $k = 1$. Therefore, we can apply the procedure done there to obtain that

$$x_0 = \sqrt{k_1 x_1^2 + k_2 x_2^2 + 2\sqrt{\lambda_{12}[-(x_1^4 + x_2^4) + I\, x_1^2 x_2^2\,]}}\,, \tag{23}$$

with λ_{12} being a function of the form

$$\lambda_{12}(k_1, k_2, I) = \frac{k_1 k_2 I + (-1 + k_1^2 + k_2^2)}{I^2 - 4},$$

and where the constants k_1 and k_2 satisfy special conditions in order to ensure that x_0 is real [11].

Expression (23) permits us to determine the general solution, $x(t)$, of any instance of family (20) in the form

$$x(t) = \sqrt{k_1 x_1^2(t) + k_2 x_2^2(t) + 2\sqrt{\lambda_{12}[-(x_1^4(t) + x_2^4(t)) + I\, x_1^2(t) x_2^2(t)\,]}}\,, \tag{24}$$

with

$$I = e^{2F(t)}(x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t))^2 + \left[\left(\frac{x_1(t)}{x_2(t)}\right)^2 + \left(\frac{x_2(t)}{x_1(t)}\right)^2\right],$$

in terms of two of its particular solutions, $x_1(t)$, $x_2(t)$, its derivatives, the constants k_1 and k_2, and the time (included in the constant of the motion I).

Note that the role of the constant I in expression (24) differs from the roles carried out by k_1 and k_2. Indeed, the value of I is fixed by the particular solutions $x_1(t)$, $x_2(t)$ and its derivatives, while, for every pair of generic solutions $x_1(t)$ and $x_2(t)$, the values of k_1 and k_2 range within certain intervals ensuring that $x(t)$ is real.

It is clear that the method illustrated here can also be applied to analyze solutions of any other family of second-order differential equations related to a Lie family by introducing the new variable $v = \dot{x}$. Additionally, it is worth noting that in the case $F(t) = 0$ the family of dissipative Milne–Pinney equations (20) reduces to a family of Milne–Pinney equations appearing broadly in the literature (see [48] and references therein), and the expression (24) takes the form of the expression obtained in [11] for these equations.

6 Conclusions and Outlook.

We have proposed a generalization of Lie Theorem in order to characterize those families of non-autonomous systems of first-order ordinary differential equations, the so-called Lie families, that admit a common time-dependent superposition rule. We have studied the relations of quasi-Lie systems and Lie families.

In order to illustrate the usefulness of our achievements, we have derived common time-dependent superposition rules for studying dissipative Milne–Pinney equations and Abel equations. In the case of Abel equations, our result expresses the general solution of any particular instance of a Lie family of non-autonomous Abel equations in terms of each generic particular solution, a constant, and the time. In this way, we have initiated a new approach to study the solutions of these equations. Additionally, it is worth noting that the analyzed Lie family of Abel equations contains an autonomous instance admitting a special kind of superposition rule derived by Chiellini [31]. Unlike such a special superposition rule, our common superposition rule does not require the use of particular solutions obeying any kind of extra condition and, therefore, it clearly represents an improvement with respect to Chiellini's technique.

We have shown how common time-dependent superposition rules can be used to analyze second-order differential equations by means of the study of a family of Milne–Pinney equations. More specifically, we have derived a common-superposition rule allowing us to obtain the general solution of any instance of such a family in terms of a generic pair of its particular solutions, their derivatives in terms of the time, and the time. Such an expression represents an interesting improvement with respect to previous results and methods, as it generalizes the superposition rule given in [11] for the usual Milne–Pinney equations to the dissipative case.

We hope to get in the future new results on the theory of common time-dependent superposition rules and, additionally, to describe new applications, where our achievements can be used.

Acknowledgements

Partial financial support by research projects MTM2009-11154 and E24/1 (DGA) are acknowledged. Research of the second author financed by the Polish Ministry of Science and Higher Education under the grant N N201 365636.

References

[1] S. Lie and G. Scheffers, *Vorlesungen uber continuierliche Gruppen mit geometrischen und anderen Anwendungen*, Teubner, Leipzig, 1893.

[2] M.E. Vessiot, *Sur une classe de d'équations différentielles*, Ann. Sci. École Norm. Sup **10**, 53–64 (1893).

[3] M.E. Vessiot, *Sur les équations différentielles ordinaires du premier ordre qui ont des systèmes fondamentaux d'intégrales*, Ann. Fac. Sci. Toulousse **1**, 1–33 (1899).

[4] A. Guldberg, *Sur les équations différentielles ordinaires qui possédent un système fondamental d'intégrales*, C.R. Acad. Sci. Paris **116**, 964–965 (1893).

[5] P. Winternitz, *Lie groups and solutions of nonlinear differential equations*, Lect. Not. Phys. **189**, 263–305 (1983).

[6] N.H. Ibragimov, *Elementary Lie group analysis and ordinary differential equations.* J. Wiley, Chichester, 1999.

[7] J.F. Cariñena, J. Grabowski and G. Marmo. *Superposition rules, Lie Theorem and partial differential equations*, Rep. Math. Phys. **60**, 237–258 (2007).

[8] J.F. Cariñena, J. de Lucas and A. Ramos, *A geometric approach to integrability conditions for Riccati equations*, Electron. J. Diff. Eqns. **122**, 1–14 (2007).

[9] R.L. Anderson, J. Harnad and P. Winternitz, *Superposition principles for matrix Riccati equations*, J. Math. Phys. **24**, 1062–1072 (1983).

[10] J.F. Cariñena, J. de Lucas and M.F. Rañada, *Recent applications of the theory of Lie systems in Ermakov systems*, SIGMA **4**, 031 (2008).

[11] J.F. Cariñena and J. de Lucas, *A nonlinear superposition rule for the Milne–Pinney equation*, Phys. Lett. A **372**, 5385–5389 (2008).

[12] M.A. del Olmo, M.A. Rodríguez and P. Winternitz, *Simple subgroups of simple Lie groups and nonlinear differential equations with superposition principles*, J. Math. Phys. **27**, 14–23 (1986).

[13] M.A. del Olmo, M.A. Rodríguez and P. Winternitz, *Superposition formulas for rectangular matrix Riccati equations*, J. Math. Phys. **28**, 530–535 (1987).

[14] J. Beckers, V. Hussin and P. Winternitz, *Complex parabolic subgroups of G_2 and nonlinear differential equations*, Lett. Math. Phys. **11**, 81–86 (1986).

[15] J. Beckers, V. Hussin and P. Winternitz, *Nonlinear equations with superposition formulas and the exceptional group G_2. I. Complex and real forms of $\mathfrak{g}_2$ and their maximal subalgebras*, J. Math. Phys. **27**, 2217–2227 (1986).

[16] J.F. Cariñena and A. Ramos, *Lie systems and connections in fibre bundles: Applications in Quantum Mechanics*, in: *Proc. Conf. Diff. Geom and Appl.*, J. Bures *et al.*, 437–452, Matfyzpress, Prague, 2005.

[17] J.F. Cariñena, J. Grabowski and J. de Lucas, *Quasi-Lie schemes: theory and applications*, J. Phys. A **42**, 335206 (2009).

[18] J.F. Cariñena and J. de Lucas, *Applications of Lie systems in dissipative Milne–Pinney equations*, Int. J. Geom. Methods Mod. Phys. **6**, 683–689 (2009).

[19] J.F. Cariñena, P.G.L. Leach and J. de Lucas, *Quasi-Lie systems and Emden–Fowler equations*, J. Math. Phys. **50**, 103515 (2009).

[20] V.M. Boyko, *Symmetry, equivalence and integrable classes of Abel equations*, in: *Symmetry and Integrability of Equations of Mathematical Physics*, Collection of Works of Institute of Mathematics **3**, 39–48, Kyiv, 2006.

[21] E.S. Cheb-Terrab and A.D. Roche, *An Abel ordinary differential equation class generalizing known integrable classes*, Eur. J. Appl. Math. **14**, 217–229 (2003).

[22] P.G.L. Leach, S.D. Maharaj and S.S. Misthry, *Nonlinear Shear-free Radiative Collapse*, Math. Methods Appl. Sci. **31**, 363–374 (2008).

[23] T. Harko and M.K. Mak, *Vacuum solutions of the gravitational field equations in the brane world model*, Phys. Rev. D **69**, 064020 (2004).

[24] P. Chauvet and J. Klapp, *Isotropic flat space cosmology in Jordan-Brans-Dicke theory*, Astrophys. Space Sci. **125**, 305–309 (1986).

[25] M. Trzetrzelewski and A.A. Zheltukhin1, *$U(1)$-invariant membranes: the zero curvature formulation, Abel and pendulum differential equations*, arXiv:0905.2095.

[26] D.E. Panayotounakos and A.B. Sotiropoulou, *On the reduction of some second-order nonlinear ODEs in physics and mechanics to first-order nonlinear integro-differential and Abels classes of equations*, Theor. Appl. Fract. Mech. **40**, 255–270 (2003).

[27] V.K. Chandrasekar, M. Lakshmanan and M. Senthilvelan, *New aspects of integrability of force-free Duffing-van der Pol oscillator and related nonlinear systems*, J. Phys. A **37**, 4527-4534 (2004).

[28] S. Esposito and E. Di Grezia, *Fermi, Majorana and the Statistical Model of Atoms*, Found. Phys. **34**, 1431–1450 (2004).

[29] S. Esposito, *Majorana Transformation for Differential Equations*, Internat. J. Theoret. Phys. **41**, 2417–2426 (2002).

[30] M. Cvetič, H. Lüb and C.N. Pope, *Massless 3-brane in M-theory*, Nuclear Phys. **613**, 167–188 (2001).

[31] A. Chiellini, *Alcune ricerche sulla forma dell'integrale generale dell'equazione differenziale del primo ordine $y' = c_0y^3 + c_1y^2 + c_2y + c_3$*, Rend. Semin. Fac. Sci. Univ. Cagliari **10**, 16 (1940).

[32] J.L. Reid and G.L. Strobel, *Nonlinear superposition rule for Abel's equation*, Phys. Lett. A **91**, 209–210 (1982).

[33] M.A.M. Alwasha, *Periodic solutions of Abel differential equations*, J. Math. Anal. Appl. **329**, 1161–1169 (2007).

[34] H.W. Chan, T. Harko and M.K. Mak, *Solutions generating technique for Abel-type nonlinear ordinary differential equations*, Comput. Math. Appl. **41**, 1395–1401 (2001).

[35] F. John, *Partial differential equations 1*, Springer-Verlag, New York, 1981.

[36] I.O. Morozov, *The Equivalence Problem for the Class of Generalized Abel Equations*, Differ. Equ. **39**, 460–461 (2003).

[37] R. Redheffer, *Steen's equation and its generalisations*, Aequationes Math. **58**, 60–72 (1999).

[38] I. Redheffer and R. Redheffer, *Steen's 1874 paper: historical survey and translation*, Aequationes Math. **61**, 131–150 (2001).

[39] J.M. Thomas, *Equations equivalent to a linear differential equation*, Proc. Amer. Math. Soc. **3**, 899–903 (1952).

[40] Y. Drossinos and P.G. Kevrekidis, *Nonlinearity from linearity: The Ermakov–Pinney equation revisited*, Math. Comp. Sim **74**, 196–202 (2007).

[41] T. Srokowski, *Position dependent friction in Quantum Mechanics*, Act. Phys. Polon. B **17**, 657–665 (1986).

[42] R.W. Hasse, *On the quantum mechanical treatment of dissipative systems*, J. Math. Phys. **16**, 2005 (1975).

[43] A.B. Nassar, *Time dependent invariant associated to Nonlinear Schrödinger Langevin Equations*, J. Math. Phys. **27**, 2949–2952 (1986).

[44] P.T.S. Alencar, J.M.F. Bassalo, L.S.G. Cancela, M. Cattani and A.B. Nassar, *Wave propagator via quantum fluid dynamics*, Phys. Rev. E **56**, 1230–1233 (1997).

[45] J. Walter, *Bemerkungen zu dem Grenzpunktfallkriterium von N. Levinson*, Math. Z. **105**, 345–350 (1968).

[46] F. Haas, *The damped Pinney equation and its applications to dissipative quantum mechanics*, Phys. Scr. **81**, 025004 (2010).

[47] J.J. Cullen and J.L. Reid, *Two theorems for time-dependent dynamical systems*, Prog. Theor. Phys. **68**, 989–991 (1982).

[48] K. Andriopoulos and P.G.L. Leach, *The Ermakov equation: A commentary*, Appl. Anal. Discrete Math. **2**, 146–157 (2008).